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Color Symmetry and Colored Polyhedra*

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Abstract

Color groups provide a partial classification of colored objects, but the coloring problem is much more complex. In this paper, criteria are established for determining whether a transitive pattern can be consistently colored with each pattern unit receiving a single color, and for calculating the number of such colorings. The colorings of the face-transitive polyhedra (a class which includes the simple crystal forms) are enumerated.

Introduction

In 1951, A. V. Shubnikov proposed a theory of color symmetry to solve the problem of coloring the faces of finite figures, such as polyhedra, with two colors, 'black' and 'white'. Each symmetry operation of the figure was associated with a unique color permutation [that is, either it mapped all the black faces onto white ones, and *vice versa*, or else it mapped each face onto a face of the same color (Fig. 1)]. In some cases this requirement, later called consistency by Loeb (1971), forced both colors to be assigned to certain faces; these faces were then considered to be 'neutral' or 'gray'. To handle this, Shubnikov proposed an 'anti-identity' operation. Alternatively, the gray faces could be subdivided into black and white sectors (Fig. 2). In the subsequent extension of the theory from two to an arbitrary number k of colors, the latter approach has generally predominated, since it fits in well with the van der Waerden–Burckhardt (1961) theory in which a k-color group $G(H)$ is determined by a subgroup H of index k in the symmetry group G.

The theory of color groups has developed rapidly in recent years, but in the meantime the problem of finding the ways in which patterns can be consistently colored with a *single* color assigned to each motif seems to have been ignored. It should be stressed that even consistently colored objects cannot be completely described or classified by color groups. On the one hand, as we have pointed out, not every object can be a realization of every color group associated with its symmetry group, unless we adopt the rather artificial strategy of subdivision. Thus, while Fig. 1 shows a satisfactory coloring of the faces of an octahedron, the

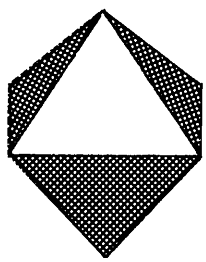

Fig. 1. The faces of the octahedron can be colored with two colors in such a way that each symmetry operation maps all faces of one color onto faces of the other color, or maps each set of single-colored faces onto itself.

* A preliminary version of this paper was presented to the XII Congress and General Assembly of the International Union of Crystallography in Ottawa, Canada, August 1981 (Senechal, 1981).

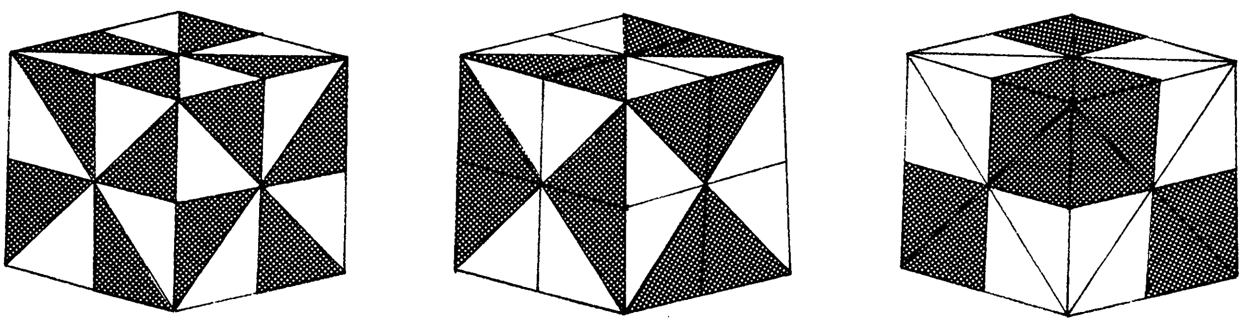

Fig. 2. In every consistent coloring of the faces of the cube with two colors, both colors must appear on each face.

diagrams in Fig. 2 may be unsatisfactory for some purposes. Another example is shown in Fig. 3, which includes two different representations of the plane group *p*6*m* as a discrete set of points. The group *p*6*m* has a subgroup of index 2 of type *p*6, and this pair defines a two-color group. But only the representation (*b*) admits a coloring described by it. Indeed, no subgroup of *p*6*m* defines a consistent two-coloring of (*a*). (Dividing the points of a pattern into asymmetric regions is even less satisfactory than subdividing the faces of a polyhedron.) On the other hand, when a pattern does admit a color group, the coloring can often be realized in more than one way. Consider the two colorings of *p*4*m* shown in Fig. 4. Both are described by the same color group but in what other sense can they represent the same physical structure? As a final example, we consider the colorings of the pyritohedron (symmetry group *m*3) associated with one of its three subgroups of type *mm*2. Since this subgroup is of index 6, $k = 6$, and there are six colors. The three inequivalent colorings of the asymmetric regions of the faces of the pyritohedron are shown in Fig. 5. We see that in (*a*) and (*b*) the faces each have one color but in (*c*) they have two. Further, the colorings in (*a*) and (*b*) cannot be considered equivalent, since in (*a*) each face is adjacent to one of the same color but in (*b*) no two faces of the same color are adjacent. The physical properties of two crystals represented by these colorings will certainly be different.

The purpose of this paper is to find criteria for predicting and distinguishing among the colorings in such figures and to enumerate the consistent colorings of face-transitive polyhedra. Although our results apply to infinite plane and three-dimensional patterns as well as to polyhedra, we restrict our attention to the latter. First we briefly review the theory of color groups. Then we show how the face-transitive polyhedra can be classified by symmetry type, and prove that this classification is the appropriate one for coloring

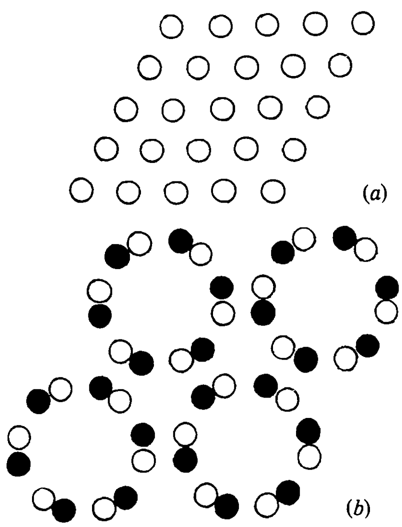

Fig. 3. In (*a*) and (*b*) we see two representations of *p*6*m* as a discrete set of points. Only (*b*) admits a coloring with two colors.

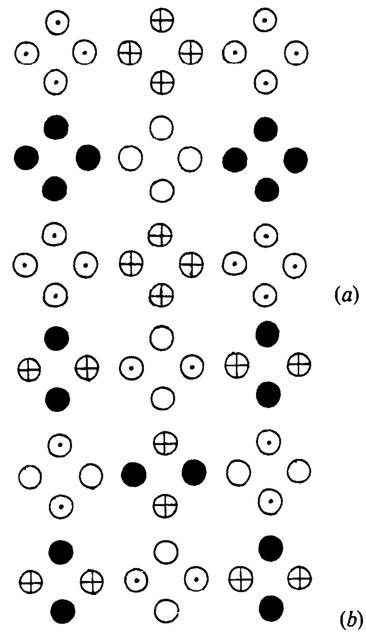

Fig. 4. These two four-colorings of *p*4*m* are described by the same color group but must be considered inequivalent.

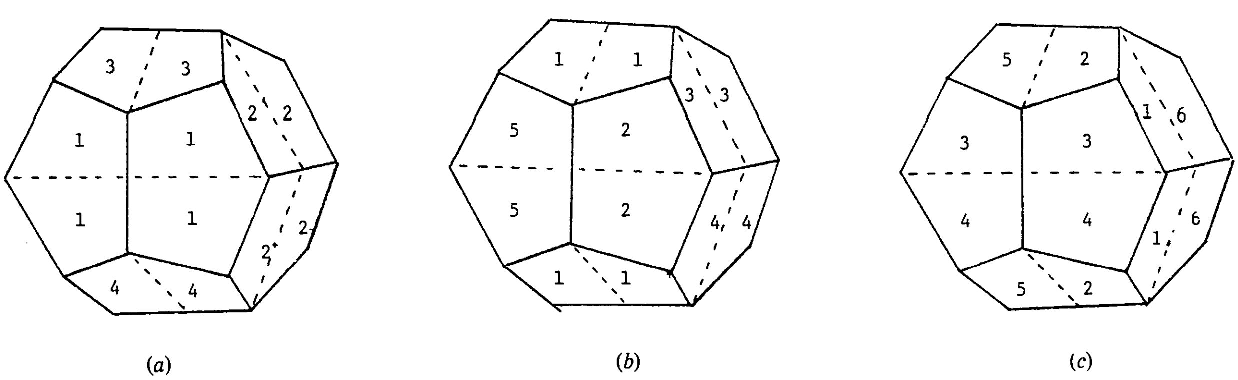

Fig. 5. There are three consistent three-colorings of the half-faces of the pyritohedron with respect to a subgroup of type *mm*2. In two of these, (*a*) and (*b*), each face is assigned a single color.

problems. Next, we enumerate the consistent colorings of the face-transitive polyhedra in which each face receives a single color. (By duality, this is also an enumeration of the consistent colorings of the vertices of the vertex-transitive polyhedra.) Just as the problem of enumerating the color groups is the problem of classifying subgroups, the problem of enumerating colored patterns is the problem of classifying the orbits of these subgroups. Finally, we briefly discuss the literature on colorings.

Color groups

The faces of a polyhedron can be divided into equivalence classes defined by the property that the faces in each class are mapped onto one another by the operations of the group. If there is only one class, the polyhedron is said to be isohedral or face-transitive. (The open and closed face-transitive polyhedra with crystallographic symmetry are the simple crystal forms.) For simplicity, by 'polyhedron' we will mean 'face-transitive polyhedron' throughout this paper.

The faces of a polyhedron with symmetry group G form an orbit for G. We recall that an orbit for a symmetry group is a set X of points, faces or other units such that

(i) the operations in the group map X onto itself, and

(ii) for any two units x_i and x_j in X, there is a symmetry operation g in the group such that $gx_i = x_j$. That is, all the elements of X are equivalent and the group acts transitively on X.

The subgroup of G which maps a given unit x_i onto itself is called the site-symmetry group or stabilizer of x_i, $S(x_i)$. For example, the stabilizer of a cube face is $4m$, the stabilizer of a face of a pyritohedron is m, and the stabilizer of a face of a gyroid is simply the identity operation 1. It is easy to show that if $gx_i = x_j$, then every operation in the left coset $gS(x_i)$ also maps x_i onto x_j, and every operation which does this belongs to that coset. (Throughout this paper group multiplication is done from right to left.) Further, $S(x_j) = gS(x_i)g^{-1}$; the stabilizers of the units of X are all conjugate. In this sense, we will speak of *the* stabilizer S of X when it is not necessary to specify a particular unit. If S contains only the identity operation 1, then X is said to be a generic or free orbit. $|X|$, the number of units in X, is equal to $|G|/|S|$; if X is a free orbit, then $|X| = |G|$. Thus the six faces of a cube form an orbit for $m3m$, in accordance with the fact that $|m3m|/|4m| = 48/8 = 6$. The gyroid's twenty-four faces form a free orbit for the group 432.

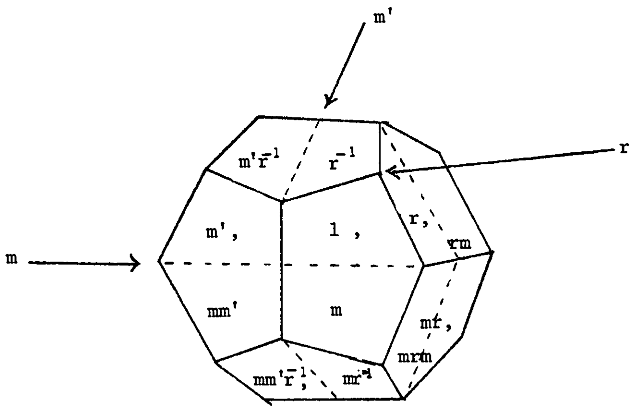

Fig. 6. The faces of the pyritohedron are an orbit for the group $m3$; the stabilizer S is of type m. Starting with one face x_1 chosen arbitrarily, we label it with the elements of $S(x_1)$ and then label the other faces with the operations in the left cosets of $S(x_1)$. (Thus the face x_1 is labeled $\{1, m\}$, where 1 is the identity element of the group $m3$.)

It is helpful to indicate the relation between a group G and an orbit X by the following device: starting with any one of X's units x_1, we label each of the others with the S operations of G which map x_1 onto it (Fig. 6). This labeling gives us an explicit diagram of the action of G on X.

With these concepts in mind, the van der Waerden–Burckhardt (1961) theory of color groups can be briefly described by the following three statements. Let G be a symmetry group and let X be a *free* orbit for G.

1. Assume that X has been consistently colored with a finite number k of colors. The set of operations in G which map a given single-color set of units onto itself is a subgroup H of G; if we label the units so that x_1 is included in this set, then the entire set is labeled with the operations H, that is, it is the set $\{hx_1, h \in H\}$, or more briefly Hx_1. An operation g which is not in H maps this set onto a set of units of some other color; those units are labeled with the operations gH. Thus each color is associated with a left coset of H and the number of colors is equal to the index of H in G.

2. Each operation g in G is associated with the color permutation

$$\begin{pmatrix} H & g_2H \ldots & g_kH \\ gH & gg_2H \ldots & gg_kH \end{pmatrix}.$$

The sets of pairs {(symmetry operation, color permutation)} is a group, called the color group $G(H)$.

3. Each subgroup H of G determines a color group as in (2), and also a consistent coloring of X: choose any unit of X to be x_1, find its images under H, and give this set of units color 1. Then label the remaining units with the operations in each of H's $k - 1$ left cosets and assign them colors 2, 3, ..., k.

As we pointed out in the introduction, to apply this coloring procedure directly to an orbit which is not free, such as the set of faces of most polyhedra, we are forced to subdivide the units into asymmetric sectors which will, in general, receive different colors. Let us say that a polyhedron P with symmetry group G admits the color group $G(H)$ if its faces can be $G(H)$-colored in such a way that each face receives a *single* color. Our first problem is to decide which polyhedra admit which color groups.

Colored polyhedra

The symmetry properties of a polyhedron are characterized by (i) its symmetry group G, (ii) the stabilizer S of its faces and (iii) the role of S in the structure of G. Thus we will consider two polyhedra P and P^* to belong to the same symmetry type if they have the same symmetry group G and if there is an affine automorphism α of G which maps the stabilizers of the faces of P onto the stabilizers of the faces of P^*. [This does not imply that P and P^* are the 'same': for example, the tristetrahedron and the deltohedron are equivalent under this classification (Fig. 7), although their combinatorial-topological properties are very different.] The face-transitive polyhedra are listed in column 1 of Table 1. In column 2 we list their symmetry groups G and the face-transitive subgroups G' of G; the latter are listed in order to include the crystallographically important case when the 'true' symmetry group is a subgroup of the symmetry group of the form. In column 3 we list the stabilizers of the faces with respect to G and G'. The corresponding symmetry type is listed in column 4, denoted by the symbols used by Grünbaum & Shephard (1981*a*,*b*) in their enumeration of the types of spherical patterns.

We are now ready to answer the question posed at the end of the last section. In Fig. 5, we noticed that in (*a*) and (*b*) the two colors of each face of the pyritohedron are the same, but in (*c*) they are different. In all three cases, H, the subgroup fixing color 1, is the same and of type $mm2$; the stabilizers of the faces of the pyritohedron are, as we noted earlier, of type m. In Figs. 5(*a*) and (*b*) the stabilizers of the faces with color 1 are subgroups of H; clearly this is necessary in order for the faces to have a single color. By inspection, we see that this requirement is not met in Fig. 5(*c*): the stabilizer of a face, one of whose colors is 1, is a subgroup for which the mirror plane is perpendicular to those in H. In general, in order for each face of a polyhedron P to have a single color, all the labels assigned to some face x_i must belong to H. This will be the case if $S(x_i) \subseteq H$. If there is no such face x_i, then some of the operations in $S(x_i)$ will always belong to different cosets of H and hence x_i will have more than one color. Thus we have

Theorem 1. A polyhedron P with symmetry group G admits the color group $G(H)$ if and only if $S(x_i) \subseteq H$ for some face x_i of P.

Further, we prove

Theorem 2. Polyhedra of the same symmetry type admit the same color groups.

To see this, let us assume that P and P^* are polyhedra of the same symmetry type, and that P admits the color group $G(H)$. Then there is a face x_i of P such that $S(x_i) \subseteq H$. By definition, there is an affine automorphism α of G which maps $S(x_i)$ onto $S(x_i^*)$, where x_i^* is some face of P^*; since α maps subgroups to subgroups it maps H onto some subgroup H^* which contains $S(x_i^*)$. Since the color groups $G(H)$ and $G(H^*)$ are equivalent (see the following section), the theorem is proved.

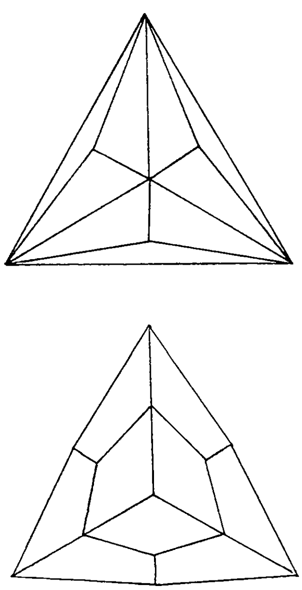

Fig. 7. The tristetrahedron and the deltohedron have the same symmetry group and stabilizer, and their faces can be matched in the required way. Thus they are of the same symmetry type.

Enumerating colored polyhedra

We now are in a position to consider the question of enumeration. Suppose that a polyhedron P admits the color group $G(H)$. When are two $G(H)$ colorings of P equivalent, and when are they distinct?

In the first place, we will assume that in order to be equivalent the two colorings must be described by the 'same' color group, that is, by equivalent color groups. The general consensus in the literature seems to be that two color groups $G_1(H_1)$ and $G_2(H_2)$ should be considered equivalent if there is an affine isomorphism between G_1 and G_2 under which H_1 and H_2 are also mapped onto one another; we will adopt this definition. For a detailed discussion see Engel & Senechal (1983). Thus the symbol $G(H)$ represents an equivalence class of color groups, although it is sometimes convenient to think of it as denoting a particular representative of this class. If $G(H_1)$ and $G(H_2)$ are equivalent, we will also say that H_1 and H_2 are equivalent.

Second, in order to determine how many distinct $G(H)$ colorings of P are possible, we must define equivalence for colorings: two $G(H)$ colorings of P, say $\bar{\bar{P}}$ and $\bar{P}$, are equivalent colorings if $\bar{\bar{P}}$ can be obtained from $\bar{P}$ by recoloring the faces of P in such a way that all the faces of $\bar{P}$ with one color are again assigned a single color, and faces with distinct colors remain

Table 1. *The face-transitive polyhedra*

In column 1 we list the polyhedra (including the open forms) whose symmetry groups act transitively on their faces; by duality, one can deduce the list of polyhedra whose symmetry groups act transitively on their vertices. In column 2 we list first the symmetry group G of the polyhedron or form, and then, when G is finite, its face-transitive subgroups G^*; their stabilizers are listed in column 3. The corresponding spherical pattern sp, numbered as by Grünbaum & Shephard (1981*b*), is given in column 4. Part A lists the open forms, part B the polyhedra with principal axes, and part C the polyhedra with cubic and isosahedral symmetry. We use the abbreviated Hermann–Mauguin notation for symmetry groups, except that we prefer rotatory-reflection, $\tilde{q}$, to rotatory-inversion, $\bar{q}$. Following Harker (1976), we use asterisks, and in some cases primes, to distinguish inequivalent operations with the same symbol. Thus m^* indicates reflection in a plane not perpendicular to a principal axis, while 2^* indicates a twofold rotation whose axis does not coincide with an axis of higher order, when one is present in the group. In columns 2, 3 and 4, q may take any positive integral values, unless otherwise indicated. The range of q for any subgroup G^* is usually the same as for the group G, so it is stated only once, unless necessary.

	G and G^*	S	sp
A. Open forms			
Pedion	∞m	∞m	3∞
Sphenoid	$mm'2$	m'	$2q, q=2$
	m	1	$1q, q=1$
	2	1	$4q, q=2$
q-gonal pyramid	qm	m	$2q, q\geq 3$
	q	1	$4q$
Di-q-gonal pyramid	qm	1	$1q, q\geq 2$
(*a*) Pinacoid	$\infty/m\ m^*$	∞m^*	$10\ \infty$
(*b*) q-gonal prism	$q/m\ m^*$	mm^*2	$9q, q\geq 3$
	q/m	m	$15q$
	qm^*	m^*	$2q$
	q	1	$4q$
	$q2$	2	$12q$
	for even q:		
	$(\frac{1}{2}q)m^*$	1	$1(\frac{1}{2}q)$
	$(\frac{1}{2}q)/m\ m^*$	m	$8(\frac{1}{2}q)$
	$(\frac{1}{2}q)\,2$	1	$11(\frac{1}{2}q)$
	$\tilde{q}m^*$	m^*	$18(\frac{1}{2}q)$
	$\tilde{q}m^*$	2	$19(\frac{1}{2}q)$
	$\tilde{q}$	1	$21(\frac{1}{2}q)$
Di-q-gonal prism	$q/m\ m^*$	m	$8q, q\geq 2$
	$\tilde{q}m^*$	1	$17q$
	qm^*	1	$1q$
	$q2^*$	1	$11q$
B. Closed polyhedra with principal axes			
q-gonal dipyramid	$q/m\ m^*$	m^*	$7q, q\geq 3$
	$q2$	1	$11q$
	q/m	1	$14q$
	$\tilde{q}m^*$, q even	1	$17q$
	$\frac{1}{2}q/m\ m^*$, q even	1	$6(\frac{1}{2}q)$
Di-q-gonal dipyramid	$q/m\ m^*$	1	$6q, q\geq 2$
(*a*) Rhombic disphenoid	222	1	$11q, q=2$
(*b*) Tetragonal disphenoid	$\tilde{4}m$	m	$18q, q=2$
	222	1	$11q, q=2$
	$\tilde{4}$	1	$21q, q=2$
Scalenohedron	$\tilde{q}m$	1	$17q, q\geq 2$
Trapezohedron	$q2$	1	$11q, q\geq 3$
Rhombohedron, trapezohedron	$\tilde{q}m$	m	$18q, q=3$, $q\geq 4$
	$q2$	1	$11q$
	$\tilde{q}$	1	$21q$

Table 1 (*cont.*)

	G and G^*	S	sp
C. Polyhedra with cubic and isosahedral symmetry			
Hextetrahedron	$\tilde{4}3m$	1	23
Tristetrahedron, deltohedran	$\tilde{4}3m$	m	24
	23	1	27
Tetrahedron	$\tilde{4}3m$	$3m$	26
	222	1	$11q, q=2$
	23	3	29
	$\tilde{4}m$	m	$18q, q=2$
	$\tilde{4}$	1	$21q, q=2$
Tetartoid	23	1	27
Hexoctahedron	$m3m^*$	1	30
Tetrahexahedron	$m3m^*$	m	31
	432	1	36
	$\tilde{4}3m$	1	23
Trisoctahedron, trapezohedron	$m3m^*$	m^*	32
	432	1	36
	$m3$	1	40
Rhombic dodecahedron	$m3m^*$	mm^*2	33
	$\tilde{4}3m$	m	24
	23	1	27
	432	2	37
	$m3$	m	41
Octahedron	$m3m^*$	$3m^*$	34
	mmm	1	$6q, q=2$
	$4/m\ mm^*$	m^*	$7q, q=4$
	42	1	$11q, q=4$
	$4/m$	1	$14q, q=4$
	$\tilde{4}m$	1	$17q, q=2$
	432	3	38
	$m3$	3	43
Cube	$m3m^*$	$4m$	35
	32	1	$11q, q=3$
	$\tilde{6}m^*$	m^*	$18q, q=3$
	$\tilde{6}$	1	$21q, q=3$
	$\tilde{4}3m^*$	$2m^*m^*$	25
	23	2	28
	432	4	39
	$m3$	$mm2$	42
Gyroid	432	1	36
Diploid	$m3$	1	40
Pyritohedron	$m3$	m	41
	23	1	27
Hexakis icosahedron	$53m$	1	44
Pentakis dodecahedron,	$53m$	m	45
triakis icosahedron, trapezoidal hexacontahedron	532	1	49
Rhombic triacontahedron	$53m$	$mm2$	46
	532	2	50
Icosahedron	$53m$	$3m$	47
	532	3	51
Pentagonal dodecahedron	$53m$	$5m$	48
	532	5	52
	$m3$	m	41
	23	1	27
Pentagonal hexacontahedron	532	1	49

distinct. Then the single-color sets of faces in one coloring are geometrically congruent to those in any equivalent coloring, although the colors themselves may differ. It is not difficult to show (Senechal, 1983) that in fact this congruence is achieved by a symmetry operation in G.

Let H be a subgroup of G which is fixed for the moment. In each $G(H)$ coloring of P we can locate a set of faces, all with the same color, which is an orbit for H, so we have a criterion for inequivalence: distinct colorings correspond to incongruent H orbits. An orbit of H is determined by the choice of the initial face x_1 of P on which the operations of H are performed. Let us first assume that the set X of faces is a free orbit, and that an H orbit, obtained by applying H to x_1, has been located. If now we choose a different starting face $x_2 = gx_1$, which is not in the first orbit, we obtain a second orbit, disjoint from the first, by applying the operations in H to x_2. Thus this second orbit consists of the faces labeled with the operations in Hg, a *right* coset of H. If the two orbits Hx_1 and Hgx_1 are congruent, then there is a g' in G which maps H onto Hg: $g'H = Hg$. Choosing g instead of g' as the representative of the left coset, we see that the orbit Hgx_1 is congruent to the orbit Hx_1 if and only if $gHg^{-1} = H$. It follows that the number of incongruent free orbits, and hence distinct colorings, is equal to the number of different conjugates of H (including H itself). For example, comparing Fig. 5 with Fig. 6, we see that the sets of half-faces labeled with color 1 in Figs. 5(*a*), (*b*) and (*c*) are the three incongruent orbits of the subgroup $mm2$.

Still letting X be a free orbit, we now count the colorings defined by subgroups equivalent to H. We obtain no new colorings from conjugate subgroups since every $G(H)$ coloring is at the same time a $G(gHg^{-1})$ coloring. To see this, we note that $gH = (gHg^{-1})g$ and hence the single-color set of units labeled with the operations of the left coset gH is precisely the orbit of gHg^{-1} which contains the unit gx_1. Further, any other single-color set g_iHx_1 of the $G(H)$ coloring is at the same time the single-color set $g_ig^{-1}(gHg^{-1})gx_1$ of the $G(gHg^{-1})$ coloring. See also Senechal (1975). On the other hand, if H_1 and H_2 are equivalent but do not belong to the same conjugacy class, then the colorings they define are distinct, and since H_1 and H_2 have the same number of conjugates there are the same number of colorings in each case. The only classes of point groups in which such subgroups occur are qm, $q2$, q/mm and $(\widetilde{2q})m$, for even q, and when they do occur the number of conjugacy classes is two (Fig. 8). [For qm, $q2$ and $(\widetilde{2q})m$ these are rm or $r2$ subgroups, where q/r is even if $r < q$; for q/mm there are also two types of r/mm subgroups if q/r is even, two types of $\tilde{q}m$ subgroups if $q/2$ is even, and two types of $\tilde{r}m$ subgroups if $q/(2r)$ is odd.] This completes the case of free orbits.

As we have seen, if the orbit is not free, then H defines a coloring of P if and only if $S(x_i) \subseteq H$ for some face x_i. The enumeration proceeds as described above, but by taking this requirement into account we delete

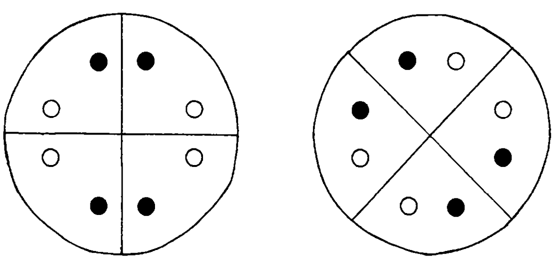

Fig. 8. The group $4m$, shown here in stereographic projection, has two equivalent but nonconjugate mm subgroups. These groups define distinct colorings.

those colorings in which more than one color is assigned to a single face, such as Fig. 5(*c*). If $x_i = gx_1$ is a face in an orbit of H labeled with the operations Hg then $S(x_i) = gS(x_1)g^{-1} \subseteq H$ or alternatively $S(x_1) \subseteq g^{-1}Hg$. Thus, to enumerate the colorings we count the number of conjugates of H which contain $S(x_1)$. Again, subgroups conjugate to H define the same colorings while equivalent but nonconjugate subgroups give an equal number of distinct colorings if the condition $S(x_i) \subseteq H$ is satisfied. (In fact, the di-q-gonal prism is the only type of polyhedron with nontrivial S which admits colorings defined by two conjugacy classes of equivalent subgroups H.)

We can summarize the above discussion succinctly:

Theorem 3. The number of $G(H)$ colorings of a polyhedron P is equal to the number of equivalent subgroups H of G which contain the stabilizer S of a given face x_1 of P.

In Table 2, we enumerate the admissible colorings of each of the polyhedral symmetry types, listing for each pair G and S the subgroups which define the classes of admissible color groups, together with the number of colors and the number of inequivalent colorings.

Comments on the literature

We conclude with a brief review of some of the literature which deals with colorings. Since van der Waerden & Burckhardt's (1961) fundamental paper first appeared, most authors have defined color groups in the way we have here, that is, a color is identified with each left coset of a subgroup H of G. It should be noted, however, that the identification of colors with cosets and, accordingly, symmetry operations with color permutations, appeared earlier in a paper by Wittke & Garrido (1959). These authors identified the colors with the right cosets of H instead of with its left cosets, and thus the colorings in their diagrams were inconsistent. A discussion of other early efforts can be found in Senechal (1975). The first author to consider the problem of enumerating colorings was Harker (1976), who defined a *diamorph* of a coloring to be a different arrangement of the same colors producing the same color group. This is a much finer classification than ours but is not designed to distinguish those colorings in which the units of a pattern receive a single color from those in which they do not. MacDonald & Street (1978) gave the definition of equivalence for colorings that we have used here, and pointed out that the number of colorings of a free orbit will be greater than the number of conjugates of H in G, if there are subgroups of G equivalent to H but not conjugate to it. However, they did not carry out any complete enumerations. Colored plane tilings are discussed by Senechal (1979); a geometric approach to their classification is proposed by Grünbaum & Shephard

Table 2. *Colorings of the spherical patterns*

In column 1 the symmetry-pattern types *sp* are numbered according to Grünbaum & Shephard (1981*b*). In columns 2, 3 and 4, we list the symmetry group G of the pattern, its stabilizer S, and the number of motifs $|X|$. In column 5 we list all the proper subgroups H of G which contain $S(x)$ for some $x \in X$, together with the number of colors $k = [G{:}H]$ and the number of c of inequivalent colorings. The notation is the same as in Table 1. [Thus for *sp* 28 the entry (222; 3,1) means that the subgroup 222 contains the stabilizer of a face, and this subgroup defines a single coloring with three colors.] The letter r always represents a proper divisor of q, or the integer 1, so q/r is always an integer; similarly, $q/2$ and $r/2$ are defined only for even q and r. When S is trivial, all subgroups H define admissible color groups. We have not computed k and c for these cases; their determination is straightforward (see text). For a list of the subgroups of the crystallographic and icosahedral groups, the reader is referred to Harker (1976).

sp	*G*	*S*	$\|X\|$	$(H{:}k, c)$
$1q$	qm	1	$2q$	all H
$2q$	qm	m	q	$(rm; q/r, 1)$
$3q$	qm	qm	1	
3∞	∞m	∞m	1	
$4q$	q	1	q	all H
$5q$	q	q	1	
$6q$	$q/m\,m^*$	1	$4q$	all H
$7q$	$q/m\,m^*$	m^*	$2q$	$(r/m\,m^*; q/r,1)$; $(qm^*; 2,1)$; $(rm^*; 2q/r,1)$; $(\tilde{q}m^*; 2,1)$, q even; $(\tilde{r}m^*; 2q/r,1)$, q/r odd, q even, r even
$8q$	$q/m\,m^*$	m	$2q$	$(r/m\,m^*; q/r$, 1 if q/r odd, 2 if q/r even), $r > 1$; $(q/m; 2,1)$; $(r/m; 2q/r,1)$; $(mm^*2^*; q,q)$
$9q$	$q/m\,m^*$	mm^*2^*	q	$(r/m\,m^*; q/r,1)$
$10q$	$q/m\,m^*$	qm^*	2	$(qm^*; 2,1)$
10∞	$\infty/m\,m^*$	∞m^*	2	$(\infty m^*; 2,1)$
$11q$	$q2^*$	1	$2q$	all H
$12q$	$q2^*$	2^*	q	$(r2^*; q/r,1)$
$13q$	$q2^*$	q	2	$(q; 2,1)$
$14q$	q/m	1	$2q$	all H
$15q$	q/m	m	q	$(r/m; q/r,1)$
$16q$	q/m	q	2	$(q; 2,1)$
$17q$	$(\widetilde{2q})m$	1	$4q$	all H
$18q$	$(\widetilde{2q})m$	m	$2q$	$(qm; 2,1)$; $(rm{:}\ 1q/r,1)$; $((\widetilde{2r})m, q/r,1)$, q/r odd; $(m; 2q,1)$
$19q$	$(\widetilde{2q})m$	2	$2q$	$(q2; 2,1)$; $(r2; 2q/r,1)$; $((\widetilde{2r})m; q/r,1)$, q/r odd; $(2; 2q,1)$
$20q$	$(\widetilde{2q})m$	qm	2	$(qm; 2,1)$
$21q$	$(\widetilde{2q})$	1	$2q$	all H
$22q$	$(\widetilde{2q})$	q	2	$(q; 2,1)$
23	$\tilde{4}3m$	1	24	all H
24	$\tilde{4}3m$	m	12	$(3m; 4,2)$; $(\tilde{4}m; 3,1)$; $(mm2; 6,1)$; $(m; 12,1)$
25	$\tilde{4}3m$	$mm2$	6	$(\tilde{4}m; 3,1)$; $(mm2; 6,1)$
26	$\tilde{4}3m$	$3m$	4	$(3m; 4,1)$
27	23	1	12	all H
28	23	2	6	$(222; 3,1)$; $(2; 6,1)$
29	23	3	4	$(3; 4,1)$
30	$m3m^*$	1	48	all H
31	$m3m^*$	m	24	$(m3; 2,1)$; $(4/m\,m^*; 3,3)$; $(\tilde{4}m; 6,2)$; $(mmm; 6,1)$; $(4mm^*; 6,2)$; $(mm^*m^*; 6,1)$; $(mm2; 12,2)$; $(mm^*2^*; 12,2)$; $(4/m; 6,1)$; $(2/m; 12,1)$; $(m; 24,1)$
32	$m3m^*$	m^*	24	$(\tilde{6}m^*; 4,2)$; $(\tilde{4}3m; 2,1)$; $(4/m\,m^*; 3,1)$; $(\tilde{4}m^*; 6,1)$; $(m^*m^*2; 12,1)$; $(4m^*m; 6,1)$; $(mm^*m^*; 6,1)$; $(mm^*2^*; 12,1)$; $(2^*/m^*; 12,1)$; $(3m^*; 8,2)$; $(m^*; 24,1)$
33	$m3m^*$	mm^*2^*	12	$(4/m\,m; 3,1)$; $(mm^*2^*; 12,1)$
34	$m3m^*$	$3m$	8	$(\tilde{4}3m; 2,1)$, $(\tilde{6}m^*; 4,1)$; $(3m^*; 8,1)$
35	$m3m^*$	$4m$	6	$(4/m\,m; 3,1)$; $(4m; 6,1)$
36	432	1	24	all H
37	432	2^*	12	$(32^*; 4,2)$; $(42; 3,1)$; $(22^*2^*; 6,1)$; $(2^*; 12,1)$

Table 2 (*cont.*)

sp	*G*	*S*	$\|X\|$	$(H{:}k, c)$
38	432	3	8	$(32^*; 4,1)$; $(3; 8,1)$
39	432	4	6	$(42; 3,1)$; $(4; 6,1)$
40	$m3$	1	24	all H
41	$m3$	m	12	$(mmm; 3,1)$; $(mm2; 6,2)$; $(2/m; 6,1)$; $(m; 12,1)$
42	$m3$	$mm2$	6	$(mmm; 3,1)$; $(mm2; 6,1)$
43	$m3$	3	8	$(23; 2,1)$; $(\tilde{6}; 4,1)$; $(3; 8,1)$
44	$53m$	1	120	all H
45	$53m$	m	60	$(\widetilde{10}m; 6,2)$; $(5m; 12,2)$; $(m3; 5,1)$; $(mmm; 15,1)$; $(mm2; 30,2)$; $(2/m; 30,1)$; $(\tilde{6}m; 10,2)$; $(3m; 20,2)$; $(m; 60,1)$
46	$53m$	$mm2$	30	$(m3; 5,1)$; $(mmm; 15,1)$; $(mm2; 30, 1)$
47	$53m$	$3m$	20	$(\tilde{6}m; 10,1)$; $(3m; 20,1)$
48	$53m$	$5m$	12	$(\widetilde{10}m; 6,1)$; $(5m; 12,1)$
49	532	1	60	all H
50	532	2	30	$(222; 15,1)$; $(32; 10,2)$; $(52; 6,2)$; $(23; 5,1)$; $(2; 30,1)$
51	532	3	20	$(32; 10,1)$; $(23; 5,2)$; $(3; 20,1)$
52	532	5	12	$(52; 6,1)$; $(5; 12,1)$

(1983). Finally, we remark that despite its title, *Symmetry groups of colored polyhedra and of colored simple crystal forms*, Kuzhukeev & Koptsik's (1978) paper has little relation to the present one; in their treatment, the faces of the polyhedra are assumed to be divided into asymmetric sectors which need not have the same color, and hence every color group is admissible.

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